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Diffusion of a passive scalar in two-dimensional turbulence

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We study the spectral statistics of the fluctuations of a passive scalar convected by a two-dimensional homogeneous isotropic turbulence using the eddy-damped quasinormal Markovian (EDQNM) theory, and – in certain cases – the near-equivalent test-field model (TFM). For zero correlation between scalar and vorticity fields it is known that these closures lead to inertial–convective ranges following respectively a k^{-1} law in the enstrophy-cascade range and a $k^{-\frac{5}{3}}$ law in the inverse-energy-cascade range. We show that the scalar cascade in the latter range is direct, and is characterized by a positive eddy diffusivity. For forced flows in which correlation between vorticity and scalar forcing is prescribed, the $k^{-\frac{5}{3}}$ range is replaced by a $k^{\frac{1}{3}}$ range if the correlation is perfect, and for imperfect correlation we describe an analysis that bridges the ranges $k^{\frac{1}{3}}$, k^{-1}.

We also examine the infrared ($k \to 0$) behaviour of the energy and scalar spectra. Statistically steady injection of energy and scalar variance at a wavenumber k_I produces an energy spectrum $E(k) \sim k^3$, and a scalar spectrum $E_\theta(k) \sim k$. In the unforced case, for any initial conditions, the energy spectrum develops a k^3 range for k less than the wavenumber characteristic of the energy-containing eddies and a k^{-3} range for larger k. This allows a demonstration – *via* closure – that this energy-bearing wavenumber decreases as t^{-1} and the enstrophy as t^{-2} (modulo logarithmic corrections), as predicted by Batchelor (1969). Finally we show the scalar-fluctuation variance decays as the enstrophy, if the enstrophy spectrum is considered as a passive scalar. If not, the decay exponent is proportional to the ratio of the characteristic eddy-damping rates of the velocity and scalar third-order moments.

1. Introduction

This paper studies the variance of the fluctuations of a passive scalar in two-dimensional isotropic homogeneous turbulence, paralleling the analysis made by Oboukhov (1949), Corrsin (1951), Batchelor (1959) and Batchelor, Howells & Townsend (1959) in the three-dimensional case. We present phenomenological arguments and analytical results obtained from non-local expansions of the transfer functions (involving elongated wavenumber triads) using the EDQNM (see e.g. Pouquet *et al.* 1975). We study the infrared (low-k) behaviour of the energy and scalar spectra and in addition the decay with time of the variance of the fluctuations.

We point out here that our present concern is strictly the study of a passive scalar such as the temperature field in two-dimensional Navier–Stokes equations. This is not the same problem as the temperature field in a quasigeostrophic system of equations (Hoyer & Sadourny 1982) – a system whose dynamics are closely similar to two-dimensional turbulence. For the latter, the temperature is not a passive scalar, but diagnostically related to the vorticity field through the 'omega' equation (for a discussion of the latter point see e.g. Charney 1971).

The paper is organized into six sections. Sections 2 recalls the main results concerning the dynamics of two-dimensional turbulence as it devolves from the EDQNM theory. Section 3 presents results concerning the k^{-1} inertial–convective and viscous–convective range in the enstrophy-cascade range. In §4 we show that the scalar flux in the inverse cascade is positive, implying a $k^{-\frac{5}{3}}$ direct inertial–convective cascade for the scalar. In §5 we consider the case of freely decaying energy and scalar variance, indicating how the present analysis supports and generalizes the earlier concepts of Batchelor (1969). Finally, in §6 we describe effects associated with correlation between the scalar and the vorticity field. We examine the case in which this correlation is injected into the statistically steady (forced) flow, and the case of decaying correlation in stationary flow, obtaining in the latter case a quantitative estimate of the rapidity of the decay of correlation.

2. Dynamics of two-dimensional turbulence

We consider two-dimensional turbulence bearing a passive scalar θ. The equations of motion for the vorticity field ξ and θ may be written as

$$\frac{\partial(\xi,\theta)}{\partial t}+\boldsymbol{u}\cdot\nabla(\xi,\theta) = \nabla^2(\nu\xi,\kappa\theta), \tag{2.1\,a,b}$$

where ν is the viscosity and κ the molecular diffusivity. The velocity $\boldsymbol{u}$ satisfies $\boldsymbol{u} = (-\psi_y, \psi_x)$, $\nabla^2\psi = \xi$. For inviscid flows (2.1) has four quadratic constants of motion: $\langle \boldsymbol{u}^2\rangle$, $\langle\xi^2\rangle$, $\langle\theta^2\rangle$ and $\langle\xi\theta\rangle$. Here the angular brackets denote a volume average over a large periodic box. We first discuss results for the case $c = \langle \xi(\boldsymbol{x})\,\theta(\boldsymbol{x}')\rangle = 0$, with $c \neq 0$ presented in §6.

Homogeneity and isotropy in the (x, y)-plane is assumed throughout this paper. The second-order statistics are then wholly characterized by the kinetic energy $E(k,t)$ and the scalar-spectrum variance $E_\theta(k,t)$, defined so that

$$\tfrac{1}{2}\langle \boldsymbol{u}^2\rangle = \int_0^\infty \mathrm{d}k\, E(k,t) \equiv \tfrac{1}{2}V^2, \tag{2.2}$$

$$\tfrac{1}{2}\langle \theta^2\rangle = \int_0^\infty \mathrm{d}k\, E_\theta(k,t). \tag{2.3}$$

Here V is the r.m.s. velocity field and $\langle\theta^2\rangle$ the volume-averaged scalar variance.

The enstrophy, defined by

$$D(t) = \tfrac{1}{2}\langle\xi^2\rangle = \int_0^\infty \mathrm{d}k\, k^2E(k,t), \tag{2.4}$$

is an inviscid constant of motion, as implied by the conservation of ξ along particle trajectories (no stretching of vortex filaments). As a result, enstrophy cascades toward small scales, and energy towards large (Fjortoft 1953).

Let ϵ and η be rates of injection of energy and enstrophy at a certain wavenumber

k_{I}, by some external force. Then for stationary flow Kraichnan (1967) and Leith (1968) predict a direct enstrophy cascade for which

$$E(k) \sim \eta^{\frac{2}{3}} k^{-3} \quad (k_{\mathrm{I}} < k < k_{\mathrm{D}}), \tag{2.5}$$

with

$$k_{\mathrm{D}} = (\eta/\nu^3)^{\frac{1}{6}}, \tag{2.6}$$

and an inverse energy cascade to large scales for which

$$E(k) \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \quad (k < k_{\mathrm{I}}). \tag{2.7}$$

These cascades may be shown to be stationary solutions of the spectral-closure equations derived from the EDQNM or the near-equivalent test-field model (TFM) (Kraichnan 1971*a, b*). Pouquet *et al.* (1975) have further shown that – in the appropriate circumstances – time-dependent calculations also converge to these stationary spectra. It must be stressed, however, that only meagre evidence – experimental or theoretical – indicates this double cascade is really the dynamics of two-dimensional turbulence (see, however, the review paper of Kraichnan & Montgomery (1979), the earlier paper of Lilly (1969) and the more recent works of Brachet & Sulem (1984) and Sulem & Frisch (1984) for some evidence for these theoretical ideas).

For freely evolving turbulence there is no inverse cascade. For this case Batchelor (1969) proposed a self-similar spectrum

$$E(k,t) = V^3 t F(kVt), \tag{2.8}$$

where F is as yet arbitrary (see also Rhines 1979). The integral scale k_{I}^{-1} is then $\sim Vt$. As noted by Tatsumi & Yanase (1981), (2.8) is then not valid in the enstrophy-dissipation range. However, (2.8) gives a contribution to the total enstrophy equivalent to that contributed by the actual enstrophy range. The enstrophy is proportional to t^{-2}, since

$$D(t) = \int \mathrm{d}k\, V^3 t k^2 F(kVt) = t^{-2} \int_0^\infty \mathrm{d}a\, a^2 F(a).$$

The enstrophy-dissipation rate is then proportional to t^{-3}, and the energy spectrum in the enstrophy-cascade range is given by

$$E(k,t) \sim t^{-2} k^{-3}. \tag{2.9}$$

Logarithmic corrections to the k^{-3} spectrum (Kraichnan 1971*a*) are described later. Such terms are simply details needed to ensure convergence of integrals, and do not alter qualitative arguments made here.

3. Scalar spectrum for $k > k_{\mathrm{I}}$

The scalar spectrum for $k > k_{\mathrm{I}}$ has been considered by Lesieur, Sommeria & Holloway (1981) and by Mirabel & Monin (1983); we shall here only briefly summarize certain results found there needed for future discussion. We assume a k^{-3} energy spectrum extending from k_{I} to k_{D}, with k_{D} given by (2.6). For $k \gg k_{\mathrm{D}}$ we assume that $E(k,t)$ decreases rapidly. Let $E_\theta(k)$ be forced at k_θ and let k_θ be in the range $k_{\mathrm{I}} < k_\theta < k_{\mathrm{D}}$. Then, as described later,

$$E_\theta(k) \propto \chi \eta^{-\frac{1}{3}} k^{-1} \quad (k > k_\theta), \tag{3.1}$$

where χ is the dissipation rate of θ.

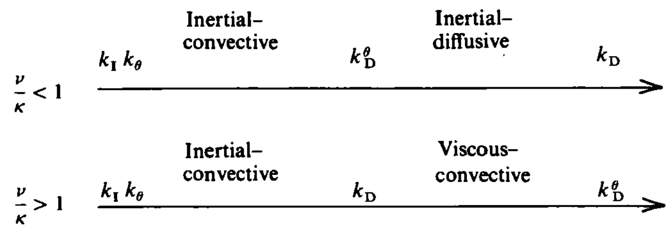

FIGURE 1. Characteristic ranges of the scalar spectrum, according to the relative values of the molecular viscosity and diffusivity.

Equation (3.1) is valid as long as molecular diffusive effects may be neglected, and for any Prandtl number $Pr = \nu/\kappa$. Equation (3.1) implies that diffusive effects become important for $k > k_D^\theta$, where

$$k_D^\theta = (\eta/\kappa^3)^{\frac{1}{6}}. \tag{3.2}$$

As in three dimensions, it is necessary to consider several different scalar spectral ranges, as summarized in figure 1. If $Pr < 1$ (implying $k_D^\theta < k_D$) and $k_\theta < k < k_D^\theta$ viscous and diffusive effects are unimportant. The scalar spectrum is 'inertial–convective'. For $k_D^\theta < k < k_D$ the velocity field is still inertial, but the diffusion of the scalar is dominated by molecular effects. The scalar spectrum is 'inertial–diffusive'. If $\nu/\kappa > 1$ ($k_D < k_D^\theta$) and $k_\theta < k < k_D$ the scalar spectrum is still 'inertial–convective'. For $k_D < k < k_D^\theta$ the scalar spectrum is 'viscous–convective'.

We now examine the inertial–convective and viscous–convective ranges in more detail.

3.1. *The inertial–convective range*

In the three-dimensional analysis of Obukhov (1949) and Corrsin (1951) the scalar spectrum is assumed to be proportional to the spectrum of the cascading quantity, i.e. the energy. If – by analogy – the cascading quantity is enstrophy, we may expect

$$E_\theta(k) \sim \frac{\chi}{\eta} E(k)\,k^2, \tag{3.3}$$

which is (3.1), on using (2.5) (also proposed by Mirabel & Monin 1982). The spectrum (3.1) may be derived via two-point closure (see the Appendix for an account of these equations). We start with the expression for the enstrophy flux $Z(k)$ through wavenumber k. This is defined by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + 2\nu k^2\right)(k^2 E) = -\frac{\partial}{\partial k} Z(k). \tag{3.4}$$

It may be shown (Kraichnan 1971*b*) from (A 1)–(A 4) of the Appendix that in the enstrophy cascade $Z(k)$ is dominated by triad interactions (k, p, q) of the form

$$q \ll p \sim k.$$

Expansion with respect to the small parameter q/k leads to

$$Z(k) = -\tfrac{1}{4}k^3 \frac{\partial k E(k)}{\partial k} \int_0^k \mathrm{d}q\, q^2 E(q)\, \tau(k, k, 0). \tag{3.5}$$

Here, $\tau(k, k, 0)$ is a triple moment relaxation time as given by (A 16) of the Appendix. (for a more complete discussion see Basdevant, Lesieur & Sadourny 1978).

For $\langle \xi(-k)\,\theta(k)\rangle = 0$, an application of the same diffusion expansion technique to the scalar gives for the scalar flux Π_θ:

$$\Pi_\theta = \int_k^\infty \mathrm{d}p\, T_\theta(p),$$

with
$$\left(\frac{\partial}{\partial t} + 2\kappa k^2\right) E_\theta \equiv T_\theta(k). \tag{3.6}$$

From (A 1), after an appropriate 'diffusion' analysis, we find

$$\Pi_\theta(k) = -\tfrac{1}{4}k^3 \frac{\partial (E_\theta(k)/k)}{\partial k} \int_0^k \mathrm{d}q\, q^2 E(q)\, \tau'(k,k,0), \tag{3.7}$$

where τ' is defined by (A 18) of the Appendix. That (3.7) is formally nearly identical to (3.5) should not be surprising upon recalling the similarity of the ξ and θ equations of motion (2.1 *a*, *b*). At small scales both ξ and θ are simply strained by the large scales; if the latter are statistically independent of small scales, variances of ξ and θ behave similarly (see e.g. Kraichnan 1975).

Equation (3.7) with (3.5) yields

$$E_\theta(k) = \frac{\chi}{\eta}\frac{\lambda'}{\lambda}\, k^2 E(k). \tag{3.8}$$

In (3.8) λ and λ' are constants that determine the triple-moment relaxation rates for $\langle \xi(x')\,\xi(x)\,\xi(x)\rangle$ and $\langle \theta(x')\,\xi(x)\,\theta(x)\rangle$ (see (A 13) of the Appendix). These numbers are not determined by the closure used here (EDMQN). However, if the enstrophy is treated as a passive scalar then

$$\lambda' = \lambda. \tag{3.9}$$

3.2. *The viscous–convective range*

For $Pr > 1$ and $k_\mathrm{D}^\theta > k_\mathrm{D}$ (3.7) remains valid, if we replace the right-hand-side integral by

$$\frac{1}{2\lambda'}\left\{\int_0^\infty \mathrm{d}q\, q^2 E(q)\right\}^{\frac{1}{2}} = \frac{1}{2\lambda'}\left(\frac{\eta}{2\nu}\right)^{\frac{1}{2}}.$$

We find, using (3.9) and assuming a constant scalar flux χ,

$$E_\theta(k) = 4\lambda\chi\eta^{-\frac{1}{2}}(2\nu)^{\frac{1}{2}}\, k^{-1}. \tag{3.10}$$

Thus in two dimensions the inertial–convective and viscous–convective ranges have the same spectral distribution, except that (3.10) has no logarithmic correction, in contrast with the inertial–convective range. Indeed, in (3.8) the logarithmic correction of the scalar spectrum is the same as the energy spectrum, as noted by Mirabel & Monin (1983). Figures 2 and 3 show schematically the energy and scalar spectra in the inertial–convective and viscous–convective ranges. It is not *a priori* certain that both k^{-1} ranges for $E_\theta(k)$ will match at k_D.

4. The scalar spectrum in the inverse-energy-cascade range

We now assume that k_θ (the scalar-injection wavenumber, or the wavenumber characteristic of the initial scalar fluctuations) lies in the $k^{-\frac{5}{3}}$ inverse energy cascade ($k_\theta < k_\mathrm{I}$). Determining the behaviour of the scalar is not *a priori* obvious. On the one hand, we expect the growth of large eddies to entrain the scalar fluctuations toward progressively larger scales. Possibly this would lead to a $k^{-\frac{5}{3}}$ inverse inertial–

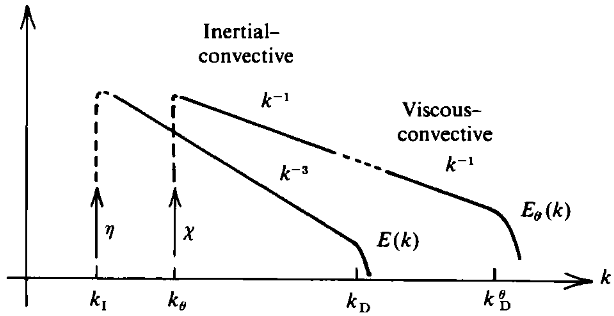

FIGURE 2. Schematic energy and scalar spectra (at $k > k_I$) for Prandtl number > 1. The enstrophy is assumed to be injected at k_I at a rate η. The scalar is injected at $k_\theta > k_I$ at a rate χ.

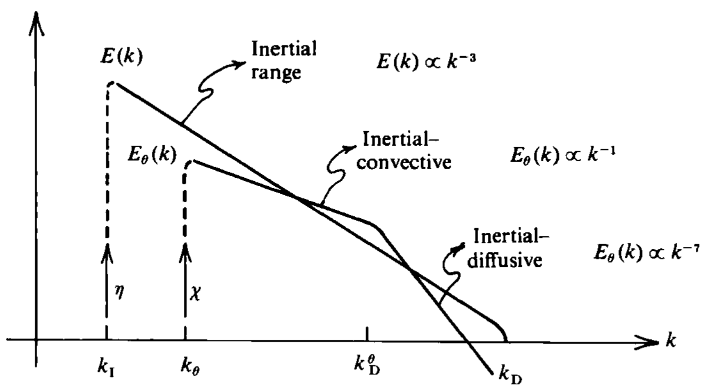

FIGURE 3. Schematic energy and scalar spectra (at $k > k_I$) for Prandtl number < 1. The k^{-7} inertial range was derived in Lesieur *et al.* (1981).

convective cascade. On the other hand, no dynamical constraint (such as the enstrophy conservation for the velocity) exists to force the large scales of the scalar field into an inverse cascade (Schertzer & Lovejoy 1984; Holloway & Kristmannsson 1984).

To provide insight into this issue we solve (A 2) numerically for $F_\theta(k) = \delta(k-k_\theta)$ and $E(k) \sim k^{-\frac{5}{3}}$, with $\langle\theta(\boldsymbol{k})\,\xi(-\boldsymbol{k})\rangle = 0$. We pick k_θ at the beginning of the $k^{-\frac{5}{3}}$ range of $E(k)$. Further details of the numerical calculation are given in the caption to figure 4. For small-k numerical stability we choose

$$E(k) = k^3(1+k)^{-\frac{14}{3}}. \tag{4.1}$$

Equation (4.1) assumes a Rayleigh friction to inhibit the development of a $k^{-\frac{5}{3}}$ range for $E(k)$ into the origin, a condition difficult to manage numerically.

Figure 4 shows the steady-state solution of (A 2) for the above forcing, and for a range of hyperconductivities as shown in the figure. We note an accurate $k^{-\frac{5}{3}}$ range extending from $k = 3$ to $k = 50$. The 'bump' in the spectrum beyond $k = 80$ actually represents a transition back to an inviscid $E_\theta(k) \sim k$ spectrum, prior to the scalar dissipation. The dissipation is actually negligible throughout this range. The inertial-range coefficient C_0 ($E_\theta = C_0(\chi/\epsilon^{\frac{1}{3}})\,k^{-\frac{5}{3}}$) may be computed from this calculation, provided we know the equivalent C_v for the velocity field. The latter has been computed by Kraichnan (1971*b*), utilizing the TFM, which is equivalent to our

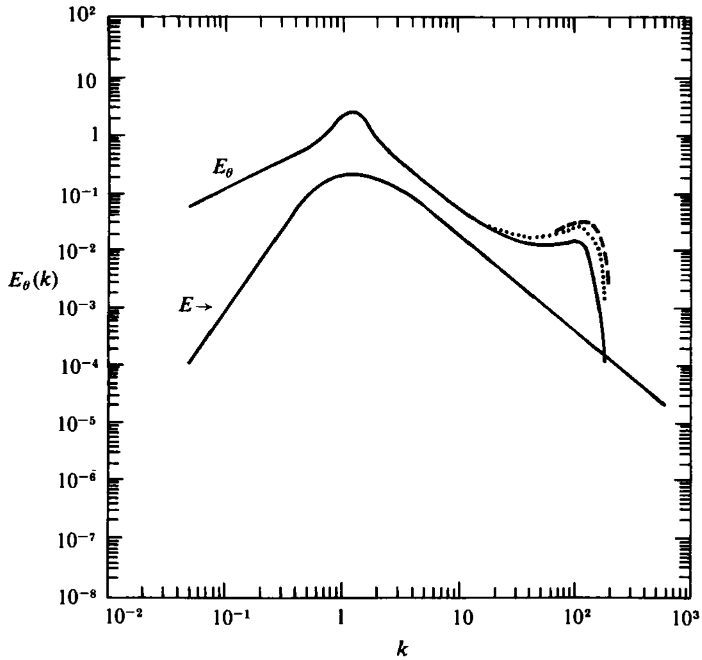

FIGURE 4. Steady-state scalar-energy spectrum $E_\theta(k)$ for $E(k)$ given by (4.1), and scalar forcing wavenumber $k_\theta = 1$. $E_\theta(k)$ is dissipated by a hyperviscosity $\kappa(k-k_H)^4$, for $k > k_H$, $k_H = 150$. The computational domain is $0 < k < 200$. A sequence of decreasing κ is shown ($\kappa = 0.004$, 0.001, 0.0005).

present EDQNM model, provided we choose $\lambda \approx 0.4$. For the scalar we must make some choice for (λ', λ''). Here we simply set these equal to λ. This gives,

$$C_0 = 0.750 C_v^{-\frac{1}{2}}. \tag{4.2}$$

For C_v Kraichnan (1971*b*) finds $C_v = 6.69$, which yields $C_0 = 0.29$. The relative smallness of this value is a consequence of the more efficient scalar cascade.

In addition to the above numerical evidence for a $k^{-\frac{5}{3}}$ scalar forward-transfer range, we offer an analysis, via a model employed by Larcheveque & Lesieur (1981). There the problem of dispersion of pairs of particles (Lagrangian tracers) is studied in two and three dimensions. Batchelor (1952) and Roberts (1961) have shown this problem may be formulated as a particular case of the passive-scalar problem: for $\kappa = 0$ the probability density $\phi(\boldsymbol{r}, t)$ that two tracers will be separated by a vector $\boldsymbol{r}$ is equal to the spacial correlation $\langle \theta(\boldsymbol{x})\, \theta(\boldsymbol{x}+\boldsymbol{r})\rangle$.

Thus the EDQNM equation ((A 2) with $c = 0$) holds for the Fourier transform of $\phi(\boldsymbol{r}, t)$. If we set $\lambda' = 0$ we can, through an inverse Fourier transform, obtain an evolution equation for $\phi(\boldsymbol{r}, t)$. In two dimensions this equation is (Larcheveque & Lesieur 1981)

$$\frac{\partial \phi(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left\{ r K_{11}(r,t)\, \frac{\partial \phi(r,t)}{\partial r}\right\}, \tag{4.3}$$

where
$$K_{11}(r,t) = 2\int_0^\infty \mathrm{d}q\, \tau'(0,0,q)\, E(q,t) \left\{1 - \frac{2J_1(qr)}{qr}\right\}; \tag{4.4}$$

here J_1 is the Bessel function of order 1. These equations are the same as obtained through the abridged LHDIA; see Kraichnan (1966), who studied the equivalent three-dimensional problem. The choice $\lambda' = 0$ may, at first, look questionable in the enstrophy-cascade range since – as we have seen in §3 – the k^{-1} inertial–convective range assumes – in principle – a non-zero value for λ'. However, Larcheveque & Lesieur's choice leads to

$$\Pi_\theta(k) = -\tfrac{1}{4}k^3 \frac{\partial(E_\theta(k)/k)}{\partial k}\int_0^\infty \mathrm{d}q\, q^2\tau'(0,0,q)\,E(q), \tag{4.5}$$

and we may verify that (3.7) and (4.5) are equivalent in the inertial–convective range if we take $\lambda' = 4\lambda''$. (The definition of the scalar relaxation parameter λ'' is given by (A 18) of the Appendix.) Thus the k^{-1} range is still recovered. Returning to the inverse $k^{-\frac{5}{3}}$ energy cascade, it is likely that the choice $\lambda' = 0$ does not fundamentally alter the dynamics of diffusion. We have examined this point numerically in three dimensions (Herring *et al.* 1982), and the results support this comment.

We may now translate the findings of Larcheveque & Lesieur (1981) in terms of the correlation $\phi(\boldsymbol{r},t)$. To this end consider an indefinite $k^{-\frac{5}{3}}$ range, in which a passive-scalar spectrum can evolve freely in time. The only constraint imposed on the scalar spectrum is that the corresponding correlation $\phi(\boldsymbol{r},t)$ should satisfy the normalization condition

$$\int \mathrm{d}\boldsymbol{r}\,\phi(\boldsymbol{r},t) = 1, \tag{4.6}$$

which stems from the fact that $\phi = \langle\theta(\boldsymbol{x})\,\theta(\boldsymbol{x}+\boldsymbol{r})\rangle$ is also a probability density distribution. Equation (4.6) implies that

$$\lim_{k\to 0}\left\{\frac{E_\theta(k)}{k}\right\} = \text{const},$$

where this constant is independent of time. This means that we restrict the small-k shape of the $E_\theta(k)$ spectrum to be of the thermal-equilibrium form for the present analysis. The assumption of an inverse energy cascade also allows the calculation of $K_{11}(r,t)$. With the aid of (4.6), we may show that (4.3) admits a similarity solution of the form

$$\phi(r,t) = R^{-2}f(r/R), \tag{4.7}$$

where the integral scale of the scalar field, defined by

$$R^2 = \int_0^\infty \mathrm{d}r\, r^2\phi(r,t), \tag{4.8}$$

satisfies a generalized two-dimensional Richardson law

$$\sigma = \frac{1}{2}\frac{\mathrm{d}R^2}{\mathrm{d}t} = \text{const}\times\epsilon^{\frac{1}{3}}R^{\frac{4}{3}}. \tag{4.9}$$

Here σ is the rate of dispersion of pairs of particles, as discussed by Batchelor (1952). Note that σ is always positive, regardless of the direction of the cascades in Fourier space. There is no evident relation between σ and scalar eddy-diffusion coefficient from initial spots as calculated for instance by Haidvogel & Keffer (1984). The non-dimensional function f in (4.7) is

$$f\left(\frac{r}{R}\right) = \text{const}\times\exp\left[-\text{const}\times\left(\frac{r}{R}\right)^{\frac{2}{3}}\right]. \tag{4.10}$$

The structure function $\phi(0,t)-\phi(r,t)$ may then be expanded for small r. It satisfies

$$\phi(0,t)-\phi(r,t) = \text{const}\times R^{-\frac{8}{3}}r^{\frac{2}{3}}. \tag{4.11}$$

We note from (4.7), (4.9) and (4.10) that

$$\chi = -\frac{\mathrm{d}\phi(0,t)}{\mathrm{d}t} \sim R^{-3}\dot{R} \sim \epsilon^{\frac{1}{3}}R^{-\frac{8}{3}}, \tag{4.12}$$

so that (4.11) becomes

$$\phi(0,t)-\phi(r,t) = \text{const} \times \chi\epsilon^{-\frac{1}{3}}r^{\frac{2}{3}}. \tag{4.13}$$

This is no more than a generalized Corrsin–Oboukhov law (Corrsin 1951; Oboukhov 1949) in the two-dimensional inverse-cascade range: the scalar flux χ given by (4.12) is positive and the scalar spectrum $E_\theta(k)$ behaves as $k^{-\frac{5}{3}}$ for $k \to \infty$, implying a direct inertial–convective cascade toward small scales.

5. Decay of passive-scalar fluctuations

In this section we consider the problem of the decay of a passive scalar, initially injected at a wavenumber k_{I}, in freely decaying two-dimensional turbulence. The energy spectrum has been studied by Batchelor (1969), Rhines (1979), Tatsumi & Yanase (1981) and Basdevant (1981), and Lesieur (1983).

We first examine the energy spectrum. Batchelor (1969) assumed – arbitrarily – a self-similar behaviour of the type (2.8). In fact, consideration of non-local transfers in the EDQNM allows us to predict completely the shape of the energy spectrum, the evolution of $k_{\mathrm{I}}(t)$, and that of the enstrophy. The derivation proceeds as follows: we first assume an enstrophy cascade of the form

$$E(k) = C\eta^{\frac{2}{3}}k^{-3}\left(1+2\ln\left(\frac{k}{k_{\mathrm{I}}}\right)\right)^{-\frac{1}{3}} \tag{5.1}$$

for $k_{\mathrm{I}} < k < k_{\mathrm{D}}$, where η is the enstrophy-dissipation rate and C is a constant. The derivation of (5.1) comes from setting $Z(k)$ in (3.5) to be constant, and hence contains only non-local transfers in the EDQNM theory. It may be extended to the freely evolving case if we suppose that the enstrophy flux is constant and equal to η in the cascade.

As $k/k_{\mathrm{I}} \to 0$ we assume

$$E(k) = C_s(t)\,k^s \quad (k < k_{\mathrm{I}}). \tag{5.2}$$

Equations (5.1) and (5.2) give a rough characterization of the k-dependence of $E(k)$, useful to give order-of-magnitude effects.

As $k \to 0$ the energy transfer may be approximated by (Kraichnan 1976; Basdevant *et al.* 1978; Herring 1978)

$$T(k) = k^3\int_0^\infty \tau(0,p,p)\,\frac{E^2(p)}{p}\,\mathrm{d}p - 2(\nu_{\mathrm{e}}+\nu(k))\,k^2E(k), \tag{5.3a}$$

with

$$\nu_{\mathrm{e}} = \begin{cases} \displaystyle\int_0^k \mathrm{d}p\left[1-\left(\frac{p}{k}\right)^2\right]^2 E(p)\,\tau(0,p,p) \quad (k_0 = 0), & (5.3b)\\[2ex] \displaystyle\frac{1}{4}\int_{k_0}^\infty \mathrm{d}p\,\tau(0,p,p)\,\frac{\partial p\,E(p)}{\partial p} \quad (k_0 > 0). & (5.3c)\end{cases}$$

The same analysis for $E_\theta(k)$ yields

$$T_\theta(h) = 2k^3\int_0^\infty \mathrm{d}p\,\tau'(0,p,p)\,E(p)\,\frac{E_\theta(p)}{p} - k^2E_\theta(k)\int_0^\infty \mathrm{d}p\,\tau'(0,p,p)\,E(p). \tag{5.4}$$

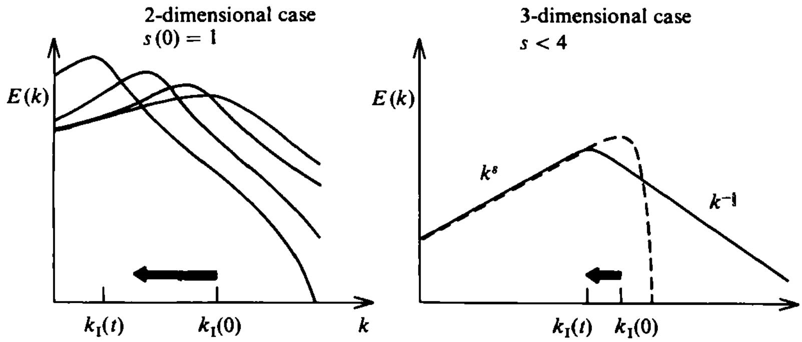

FIGURE 5. Time evolution of initial spectra peaked at $k_I(0)$. Initially these decrease rapidly above $k_I(0)$, and are proportional to k^s for $k < k_I(0)$. The three-dimensional case (Lesieur & Schertzer 1978) has $s < 4$, while $s(0) = 1$ for two dimensions. If the latter condition (for two dimensions) were to be continued for $t > 0$ the integral scale could not increase while conserving the energy exactly. Instead, however, $s \to 3$ with more rapid approach at larger k. In the three-dimensional case, and for $s > 4$, the EDQNM predicts that $s(t) \to 4$ for increasing t.

ν_e is the eddy viscosity, which is negative for problems with a lower cutoff wavenumber $k_0 > 0$, and for any spectrum decreasing at sufficiently large k faster than k^{-1}. Note that the eddy diffusivity implied by (5.4),

$$D_e = \frac{1}{2}\int_0^\infty \mathrm{d}p\,\tau'(0,p,p)\,E(p), \tag{5.5}$$

is always positive for any k_0. The first term on the right-hand-side of (5.3) gives a k^3 spectrum at low k, at least for moderate times after $t = 0$. It then is tempting to argue by analogy to three dimensions that an initial spectrum for which $E(k) \to k^s$, $k \to 0$, would be maintained (with $\mathrm{d}C/\mathrm{d}t = 0$) for $s < 3$. But this cannot be so here, because of the constraints imposed by the inviscid conservation of energy and enstrophy. Note first that the contributions of (5.2) and (5.1) to the kinetic energy are of the same order $\sim C_s\,k_I^{s+1}$. Then time-independence of C_s implies that k_I is also fixed with time, in contradiction with Fjortoft's (1953) result. The only way of resolving the contradiction is to assume that in the neighbourhood of k_I ($k \lesssim k_I$) the transfer will be strong enough to produce a k^3 spectrum that overshadows the k^s spectrum. Figure 5 shows (TFM) numerical results illustrating this point. For comparison, the three-dimensional case studied by Lesieur & Schertzer (1978) is also shown.

From this analysis we deduce that, whatever the initial spectrum is for low k, it will in time develop a k^3 range for $k \lesssim k_I$ and the initial k^s shape is relegated to an insignificant zone near $k = 0$. Equation (5.2) must then be changed to

$$E(k) = C_3(t)\,k^3 \quad (k < k_I).$$

In evaluating the total enstrophy, the contribution from the k^3 range is negligible compared with that from the enstrophy cascade (the integral of k^2 times (5.1)). Then for the total enstrophy $D(t)$,

$$D(t) \approx 3\times 2^{-\frac{4}{3}} C\eta^{\frac{2}{3}}\left(\ln\frac{k_D}{k_I}\right)^{\frac{2}{3}}. \tag{5.6}$$

If we neglect the time dependence of the logarithmic correction in (5.6) we find

$$D(t) \sim C^3\left(\ln\frac{k_D}{k_I}\right)^2 (t-t_0)^{-2}, \tag{5.7}$$

where t_0 is a virtual origin of time. This allows us to determine η. Then, by evaluating the total kinetic energy $\frac{1}{2}V^2$ and matching the k^3 and k^{-3} ranges of the spectrum at $k = k_{\rm I}$, we finally obtain

$$k_{\rm I} \sim C^{\frac{3}{2}} \left(\ln \frac{k_{\rm D}}{k_{\rm I}}\right)^{\frac{2}{3}} V^{-1}(t-t_0)^{-1}, \tag{5.8}$$

$$C_3 \sim C^{-6} \left(\ln \frac{k_{\rm D}}{k_{\rm I}}\right)^{-\frac{8}{3}} V^6(t-t_0)^4. \tag{5.9}$$

Thus Batchelor's self-similar two-dimensional energy spectrum is derived (modulo some logarithmic corrections) from an analysis of nonlocal expansions in the EDQNM theory.

We next turn to the evolution of the passive-scalar spectrum initially peaked at k_θ, considering a decaying scalar spectrum of the (approximate) form

$$E_\theta(k) = \begin{cases} C_s^\theta(t)\, k^s \quad (k < k_\theta), & (5.10a) \\ \dfrac{\lambda'}{\lambda}\dfrac{\chi}{\eta}\, k^2 E(k) \quad (k > k_\theta), & (5.10b) \end{cases}$$

where (5.10b) is just (3.15). For simplicity, we take the initial 'scalar integral scale' $(k_\theta(0))^{-1}$ to be $\sim (k_{\rm I}(0))^{-1}$. But it is not clear that $k_\theta(t)$ and $k_{\rm I}(t)$ remain of the same order as time proceeds: the strong dynamical constraint that determines the large scales of the velocity does not exist for the scalar. In particular, the k^3 scalar transfer at $k \rightarrow 0$ may not be strong enough to overwhelm an initial k^s $(s < 3)$ scalar spectrum. However, the precise infrared behaviour of the scalar spectrum and location of $k_\theta(t)$ are not needed to determine the time decay of the scalar fluctuations. Equation (5.10) yields

$$\langle \theta^2 \rangle \approx \frac{\lambda'}{\lambda}\frac{\chi}{\eta} D(t), \tag{5.11}$$

with a correction which can be shown to be negligible if the Reynolds number is high. (This is the same condition that allowed us to neglect the infrared contribution of the energy spectrum to the enstrophy.) Then (5.11) may be rewritten as

$$\frac{1}{\langle \theta^2 \rangle}\frac{{\rm d}\langle \theta^2 \rangle}{{\rm d}t} = \frac{\lambda}{\lambda'}\frac{1}{D}\frac{{\rm d}D}{{\rm d}t}. \tag{5.12}$$

Consequently
$$\langle \theta^2 \rangle = {\rm const} \times D^{\lambda/\lambda'} \propto (t-t_0)^{-2\lambda/\lambda'}. \tag{5.13}$$

As remarked in §3, it seems plausible that $\lambda' = \lambda$, which would yield a t^{-2} decay law for $\langle \theta'^2 \rangle$.

6. Correlation of scalar with velocity field

We remarked earlier that the correlation of scalar and vorticity is an inviscid constant of motion, and this implies that an initially perfect correlation is preserved in time if $Pr = 1$, and the forcing of the scalar and vorticity are the same. Here we examine the closure's prescription for correlation, $c(k) = \langle \theta(\boldsymbol{k})\, \xi(-\boldsymbol{k}) \rangle$. Our equations for $c(k)$ are quite similar to those proposed by Holloway & Kristmannsson (1984), whose point of view – with respect to theory – is closer to the Langevin equations proposed by Leith (1971). The equations for the set (E, E_θ, c) are (A 1), (A 2) and (A 3) of the Appendix. Our derivation here parallels the original derivation of the direct-interaction approximation by Kraichnan (1959).

For these equations, we may verify inviscid equipartitioning constants of motion

$\int \mathrm{d}\boldsymbol{k}\,\Theta$, $\int \mathrm{d}\boldsymbol{k}\,U(k)$, $\int \mathrm{d}\boldsymbol{k}\,U(k)/k^2$ and $\int \mathrm{d}\boldsymbol{k}\,c(k)$. (We remind the reader that $U(k)$ is here the modal enstrophy spectrum.) The equipartition solution is ($c(k) = \rho U(k)$, $U(k) = k^2/(A+Bk^2)$), $\theta = CU(k)/k$ where (ρ, A, B, C) are constants.

We notice that (A 3) is a linear equation in c; given $E(k)$ we may solve it and then substitute that solution into (A 3), where the effects of $c(k)$ serve as an additional inhomogeneity on Θ. The special case $F_c(k) = F_\zeta(k)/\rho$, $Pr = 1$ may be readily solved in the steady state. In this case

$$c(k) = \rho U(k). \tag{6.1}$$

Substituting (6.1) into the steady-state form of (A 3) we find – after using properties of $B(k,p,q)$ and $B^\theta(k,p,q)$

$$\kappa k^2[\Theta(k) - \rho^2 U(k)] = k^2(1-\rho^2)\,F_\zeta(k) + J(U, \Theta(k) - p^2 U). \tag{6.2}$$

Here J is the modal form of the scalar transfer as defined by (3.6):

$$J = \frac{1}{2\pi k}\,T_\theta\left(\frac{kE}{\pi}, \frac{E_\theta}{2\pi k}\right). \tag{6.3}$$

Thus the field $E_\theta(k) - k^2 E(k)$ is an uncorrelated scalar with a forcing $k^2(1-\rho^2)\,F(k)$. If we denote by Θ the solution of (6.2) with unit forcing, we may readily work out the normalized correlation coefficient $x(k) = c(k)\,[U(k)\,\Theta(k)]^{\frac{1}{2}}$. Assuming $F(k) \sim \delta(k-k_0)$, we find, for example,

$$x(k) = \rho U(k)\,[U(k)\,(k_0^2(1-\rho^2)\,\Theta(k) + \rho^2 U(k)]^{-\frac{1}{2}}. \tag{6.4}$$

Note that where $\Theta(k) \ll U(k)$, $x(k) \rightarrow 1$. Typically this can occur at small k, where $\Theta(k) = 1$ (thermal-equilibrium form), if $U(k)$ is inverse-cascading.

Equation (6.2) derives from two simple facts. First, under the conditions of its derivation, any linear combination $\alpha\xi + \beta\theta$ satisfies the linear passive-scalar equation, since both ξ and θ do separately. Secondly, (6.1) is valid, according to (A 3). Note that this is simply a generalization of the thermal-equilibrium relation into the forced dissipative domain. Then it follows that there is a linear combination of (ξ, θ) (i.e. Θ) uncorrelated with ξ. That such thermal-equilibrium relationships (6.1) may have a validity in steady-state forced dissipative conditions has been frequently stressed by Holloway (1985).

There remains for us to discuss the rapidity with which an injected correlation between θ and ξ disappears if there is no injection mechanism for c ($F_c(k) = 0$). The linearity of (A 3) implies that this is the same problem as the relaxation of $c(k,t)$ back to its (forced) equilibrium if initially disturbed. The key problem is then to examine the eigenmode of $(\kappa+\nu)\,k^2 - \boldsymbol{X}$, where $\boldsymbol{X}$ is a matrix representation of the right-hand side of (A 3). We anticipate that all eigenvalues Λ have negative real parts. The largest Λ_{L} then forms the basis for discussing the relaxation of correlations.

These eigenvalues are readily computed (at least numerically) provided we make an adiabatic evaluation of the relaxation effects in (A 16), putting transient factors $1 - \exp(-\Sigma\mu t) = 1$. To fix ideas, we consider the case $Pr = 1$, with the following specification of forcing and dissipation:

$$F(k) = F_0\,\frac{\delta(k-k_0)}{2\pi k} = \frac{F_\theta}{k^2} = \frac{F_c}{(\rho k)^2}, \tag{6.5}$$

$$\nu(k)\,k^2 = \kappa(k)\,k^2 = \frac{ak_1}{k+k_1} + b(k-k_1)_+^4. \tag{6.6}$$

In (6.6) our notation is that $(x)_+ = x$ $(x > 0)$ and $= 0$ $(x < 0)$. The presence of a

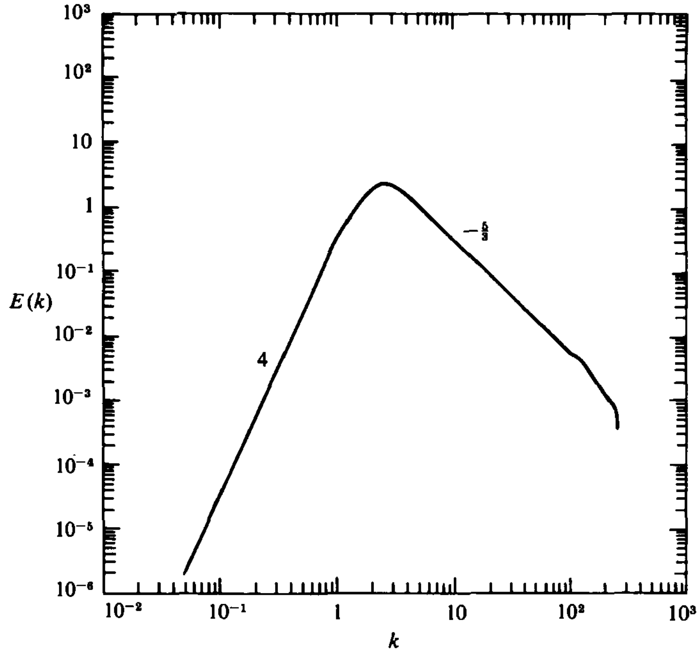

FIGURE 6. Steady-state kinetic-energy spectrum $E(k)$ with forcing and dissipation as specified by (6.5) and (6.6). The k^4 range at small k is a balance between the input term (of (5.3*a*)), and $\nu(k)$ $E(k)$, with $\nu(k)$ given by (6.6).

Rayleigh friction ($a \neq 0$) in (6.6) arrests the inverse cascade near $k = 0$. The calculations reported here all have $a = 20$, $b = 6 \times 10^{-8}(k_T/(k_T - k_0))^4$, $k_1 = 0.048$, $k_T = 256$, $k_L = 225$, $k_0 = 130$ and $F_0 = 8$. The steady-state solution for $E(k)$ is shown in figure 6. We may distinguish several spectral regions: (1) a k^4 'eddy-viscous' range, which follows from the steady-state $k \to 0$ asymptotics if we use (6.6); (2) a $k^{-\frac{5}{3}}$ range extending over $3 < k < 130$; (3) an abbreviated k^{-3} range extending over $130 < k < 225$; and, finally, (4) a dissipation range ($225 < k < 256$). To have some measure on the role of nonlinearity, it is convenient to introduce an energy-scale Reynolds number *Re* by defining an 'effective' viscosity,

$$\hat{\nu} = \frac{\int_0^\infty dk\, \nu(k)\, k^2 E(k)}{\int dk\, k^2 E(k)}, \tag{6.7}$$

$$Re = \frac{V^{\frac{1}{2}}}{\hat{\nu}} \frac{\int_0^\infty (dk/k)\, E(k)}{\int_0^\infty dk\, E(k)}. \tag{6.8}$$

An appropriate rate to which the eigenvalues may be compared is

$$\eta_{\text{inj}} = \frac{F_0}{\int_0^\infty dk\, E(k)}. \tag{6.9}$$

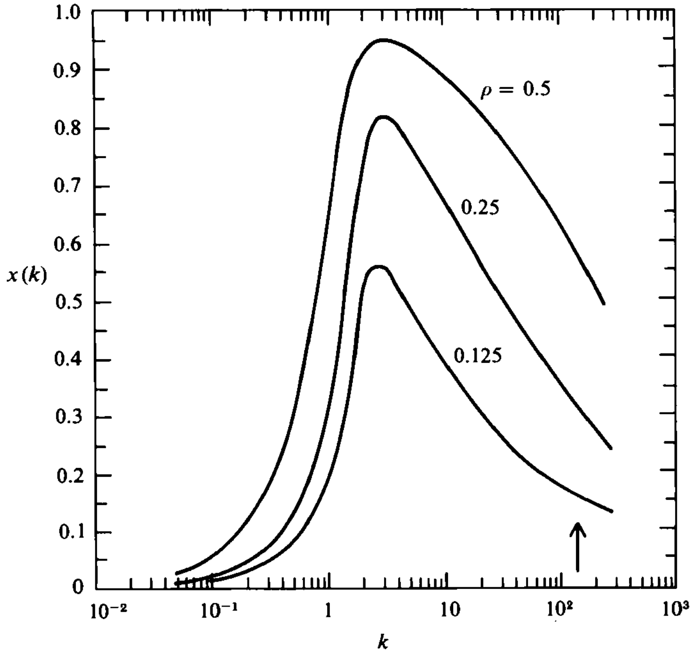

FIGURE 7. Normalized correlation coefficient $x(k)$ as given by (6.4), for correlation forcing $\rho(k)\,F(k)$; $F(k)$ is given by (6.5) and $\kappa(k)$ by (6.6). Curves for $\rho = 0.125$, 0.25, 0.5 are shown. Arrow indicates injection wavenumber.

For our present calculation $Re = 166$, which gives an extensive $k^{-\frac{5}{3}}$ range, as may be seen from figure 6. This value of Re is similar to estimates for the planetary scales of the atmosphere. For $E(k)$ as given by figure 6 we find

$$\Lambda_L = -5.24\eta_{\text{inj}}. \tag{6.10}$$

Thus correlation will disappear rather rapidly in forced dissipative system if it is not continuously injected.

Figure 7 shows steady-state $x(k)$, the normalized correlation coefficient for values of $\rho = (0, 0.125, 0.25, 0.5)$. We note that $x(k)$ increases above its injection value ρ as k decreases, until the eddy-viscous range is reached, after which it rapidly decreases. The scalar spectra $E_\theta(k)$ are shown in figure 8 for the same range of ρ. The case $\rho = 0$ has an extensive equipartition $E_\theta(k) \sim k$ range that follows the eddy-diffusive range $(3 < k < k_0)$. For the remaining curves $E_\theta(k) \sim k^{\frac{1}{3}}$ in the above range. The $E_\theta(k) \sim k^{-1}$ range (beyond k_I) for these calculations is not very well defined. The scalar 'bump' – discussed at the end of §4 – contaminates a good part of this range, making the $E_\theta(k)$ too shallow.

7. Conclusions and perspectives

This paper has examined the (second-order) statistics of the fluctuations of a passive scalar convected by two-dimensional turbulence. Using the tools of the EDQNM we have discussed four ranges for the scalar variance: (1) a k^{-1} inertial–convective range in the enstrophy cascade; (2) a k^{-1} viscous–convective range in the enstrophy-dissipation range (at high Prandtl numbers); and (3) a $k^{-\frac{5}{3}}$ inertial–convective range in the inverse-energy-cascade range (in the forced case only). In the

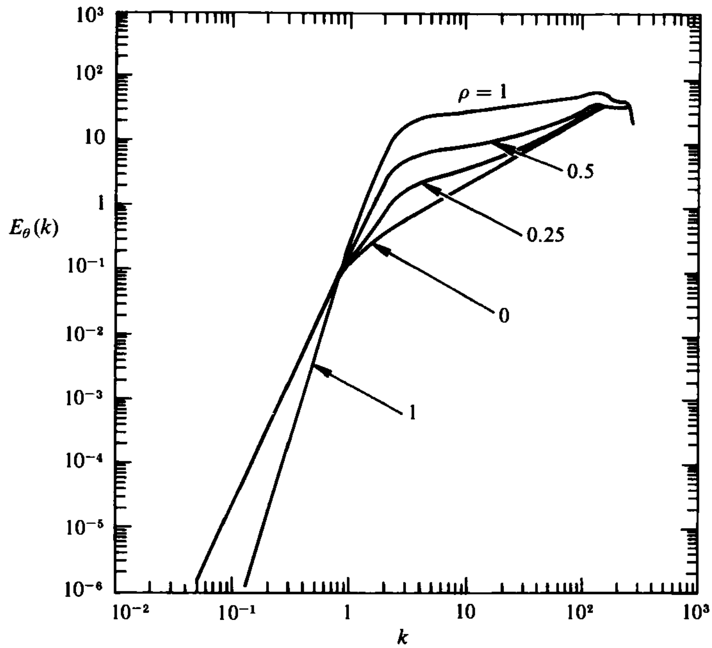

FIGURE 8. Scalar energy spectra $E_\theta(k)$ for $\rho = 0, 0.25, 0.5, 1$ and forcing and dissipation as in figure 6.

latter, the scalar cascades toward larger wavenumbers. We have discussed this point analytically and numerically.

In order to compute an eddy diffusivity we have studied also the transfer due to very elongated triads of wavenumbers: in the inertial–convective range the dynamics of the scalar is the same as the vorticity. But the eddy diffusivity is always positive. For the energy the eddy viscosity may be negative if the lowest available scale is not zero. This implies, for the case of a velocity and scalar forcing, an infrared equipartitioning spectrum for the scalar. We also examined the case of freely evolving turbulence; a non-local analysis of the statistical theory shows that an arbitrary energy spectrum evolves toward the self-similar shape predicted by Batchelor (1969) with an integral scale increasing proportionally to the time. The scalar fluctuations then decay as t^{-2}, as does the enstrophy (if one accepts that the enstrophy fluctuations behave like a passive scalar in small scales). Finally, we have examined the generalizations needed to study the case in which correlations exist between the vorticity and scalar fields. For statistically steady flows maintained by random stirring at a particular wavenumber, the scalar spectrum for perfect vorticity correlation must of course be $\sim k^{\frac{1}{3}}$ in the inverse-cascade range. Such a spectrum has been occasionally mentioned in the oceanographic context, and we see here *via* closure how it may be systematically generalized to the case in which the correlation is imperfect. If not maintained by steady injection such correlations quickly decay. The reasons, as pointed out by Holloway & Kristmannsson (1984), lie in the fact that the vorticity is more rigorously constrained by its double conservation law than the scalar, which has only one.

Several issues are left unanswered. First, on the purely technical level, some of the uncertainties concerning the behaviour of the scalar integral scale could be clarified by numerical calculations of the spectral equations (3.4)–(3.7). Considering the fragility of closure theory, certain of the above results should also be examined using direct numerical simulations of the Navier–Stokes equations. Indeed, progress here has been made through the calculations of Holloway & Kristmannsson (1984); however, we should note that much higher resolution is needed to settle issues pertaining to the integral scales of the flow. In this regard, a particularly sensitive assumption made to deduce the $\langle\theta(t)^2\rangle \sim t^{-2}$ is that the vorticity at very small scales is analogous to a passive scalar (see (3.17)).

Throughout this paper, we have traced several analogies between the scalar and vorticity fields, as indeed have several previous authors. We should stress, however – as have Holloway & Kristmannsson (1984) – that dynamically these fields behave completely differently, except in the special circumstances that at $Pr = 1$ they have the same initial data (in the spin-down problem) or have the same forcing function (in stationarily maintained flow). Indeed, we have seen in §6 that any initial correlation between them rapidly decays, if not continuously injected.

On the other hand, these dynamical differences do not necessarily vitiate the small-scale analogy between these quantities, if their spectra decrease rapidly enough. The latter is simply a necessary condition for small-scale statistical independence of large-scale (strain), as postulated by Kraichnan (1975) in making this proposal. We should note however that such statistical independence has recently been questioned by Babiano *et al.* (1984).

A final comment seems appropriate concerning the question of whether the closure theory studied here is a secure basis for sensible investigations of two-dimensional turbulence. Our position here has been that a knowledge of the consequences of the closure is useful first-order information. The closure asserts a near-Gaussianity for the flow's dynamics, and the consequences of such assumptions may be useful if for no other reason than to form an assessable basis to compare reality. On a simpler plane, this is why skewnesses and kurtosis are useful in discussing experimental flows. Recent numerical studies have suggested that purely two-dimensional flows – in certain circumstances – tend toward a system of isolated vortices, whose dynamics are not encompassed by the present methods (Basdevant & Sadourny 1983; Herring & McWilliams, 1985). The most acute departure of the real dynamics from that described here occurs for spin-down problems at long times, after the enstrophy dissipation maximum is long past. On the other hand, simulation results for stationary randomly forced two-dimensional flows are in much better accord with theory. Hence our results may still have a degree of realism for stationary turbulence.

Appendix

The equations for the modal enstrophy and temperature variance spectra, defined by

$$U(k,t) \equiv \frac{kE(k)}{\pi}$$

$$\Theta(k,t) \equiv \frac{E_\theta(k,t)}{2\pi k},$$

$$c(k,t) \equiv \langle \zeta(-\boldsymbol{k})\,\theta(\boldsymbol{k})\rangle,$$

are (in the direct-interaction approximation);

$$\left(\frac{1}{2}\frac{\partial}{\partial t}+\nu k^2\right)U(k,t) = \int_\Delta \mathrm{d}p\,\mathrm{d}q\,B(k,p,q)\,\{\hat{G}_\zeta(k)^*\,\hat{U}(p)-\hat{U}(k)^*\,\hat{G}_\zeta(p)\}\,\hat{U}(q), \quad \text{(A 1)}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial t}+\kappa k^2\right)\Theta(k,t)$$
$$=\int_\Delta \mathrm{d}p\,\mathrm{d}q\,B^\theta(k,p,q)\left[\hat{G}_\theta(k)*\left\{\hat{\Theta}(p)\,\hat{U}(q)-\left(\frac{q}{p}\right)^2\hat{c}(p)\,\hat{c}(q)\right\}-\hat{\Theta}(k)*G_\theta(p)\,\hat{U}(q)\right]$$
$$-\int_\Delta \mathrm{d}p\,\mathrm{d}q\,B(k,p,q)\,\hat{G}_\zeta(p)*\hat{c}(k)\,\hat{c}(q)$$
$$+\int_\Delta \mathrm{d}p\,\mathrm{d}q\,B^\theta(k,p,q)\left\{\hat{G}_\zeta(p)+\left(\frac{q}{k}\right)^2(\hat{G}_\theta(p)-\hat{G}_\zeta(p))\right\}*\hat{c}(k)\,\hat{c}(q)+F_\theta(k), \tag{A 2}$$

$$\left(\frac{\partial}{\partial t}+(\kappa+\nu)\,k^2\right)c(k,t)$$
$$=\int_\Delta \mathrm{d}p\,\mathrm{d}q(p^2-q^2)\,\frac{B^\theta(k,p,q)}{p^2}\{\hat{G}_\theta(k)+\hat{G}_\zeta(k)\}*\hat{c}(p)\,\hat{U}(q)$$
$$-\int_\Delta (B(k,p,q)\,\hat{G}_\zeta(p)*+B^\theta(k,p,q)\,\hat{G}_\theta(p)*)\,\hat{c}(k)\,\hat{U}(p)\,\mathrm{d}p\,\mathrm{d}q$$
$$+\int_\Delta (B^\theta(k,p,q)\,\hat{G}_\zeta(p)*-B(k,p,q)\,\hat{G}_\zeta(p)*)\,\hat{U}(k)\,\hat{c}(q)\,\mathrm{d}p\,\mathrm{d}q$$
$$+\int_\Delta B^\theta(k,p,q)\left(\frac{q}{k}\right)^2(\hat{G}_\theta(p)-\hat{G}_\zeta(p))*\hat{U}(k)\,\hat{c}(q)\,\mathrm{d}p\,\mathrm{d}q+F_c(k), \tag{A 3}$$

$$B(k,p,q)=2(k^2-q^2)\,(p^2-q^2)\,(1-x^2)^{\frac{1}{2}}/(k^2q^2), \tag{A 4}$$

$$B^\theta(k,p,q)=2\left(\frac{p}{q}\right)^2(1-x^2)^{\frac{1}{2}}. \tag{A 5}$$

In (A 1)–(A 3) we use the abbreviated notation

$$f(k)*g(p)=\int_0^t \mathrm{d}s\,f(k,t,s)\,g(p,t,s), \tag{A 6}$$

with
$$F_\theta(k)=\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{d}s\langle f_\theta(\boldsymbol{k},t)\,f_\theta(-\boldsymbol{k},s)\rangle, \tag{A 7}$$

$$F_c(k)=\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{d}s\langle f_\zeta(\boldsymbol{k},t)\,f_\theta(-k,s)+f_\theta(\boldsymbol{k},t)\,f_\zeta(-\boldsymbol{k},s)\rangle, \tag{A 8}$$

$$\hat{U}(k)=U(k,t,s)=\langle\xi(\boldsymbol{k},t)\,\xi(-\boldsymbol{k},s)\rangle, \tag{A 9}$$

$$\hat{\Theta}(k)=\Theta(k,t,s)=\langle\theta(\boldsymbol{k},t)\,\theta(-\boldsymbol{k},s)\rangle. \tag{A 10}$$

The EDQNM used here may be considered an abridgement of the DIA in which the ensemble-mean vorticity and scalar Green functions G_θ and G_ζ in (A 1)–(A 3) are approximated as

$$G_\theta(k,t,s)=\exp(-\mu'(t-s)), \tag{A 11}$$

$$G_\zeta(k,t,s)=\exp(-\mu(t-s)), \tag{A 12}$$

with
$$(\mu,\mu',\mu'')=(\lambda,\lambda',\lambda'')\left\{\int_0^k \mathrm{d}p\,p^2E(p)\right\}^{\frac{1}{2}}+(\nu,\kappa,\nu)\,k^2. \tag{A 13}$$

The EDQNM further takes the two-time correlations as

$$U(k,t,s) = U(k,t,t)\exp(-\mu(k)(t-s)), \tag{A 14}$$

$$\Theta(k,t,s) = \Theta(k,t,t)\exp(-\mu'(k)(t-s)). \tag{A 15}$$

The forcing functions F_θ and F_c assume white-noise forcing for (f_θ, f_ζ).

The EDQNM incorporates invariance to random Galilean transformation by using both a Markovianization of two-time quantities (A 14) and (A 15) and by using a Lagrangian-like Green function (A 11) and (A 12), instead of Eulerian Green functions as in the DIA. We do not record the latter here.

With (A 11)–(A 15) the time integrals (A 6) in (A 1)–(A 3) may be performed, with the result that the $B(k,p,q)$-factors are modulated by factors like

$$\tau_{kpq} = \frac{1-\exp[-t(\mu_k+\mu_p+\mu_q)]}{\mu_k+\mu_p+\mu_q}. \tag{A 16}$$

An alternative derivation of spectral-closure equations proceeds *via* an eddy-damping *Ansatz*, which represents the damping of triple cumulants caused by effects of fourth cumulants (for a more detailed discussion see e.g. Larcheveque & Lesieur 1981). As applied to the passive-scalar problems, this method gives a more general τ'_{kpq}, which contains the additional parameter λ'' (defined by (A 13)). For the uncorrelated case ($c(k) = 0$), the EDQNM procedure results in parameterizing (in (A 2)) $\hat{G}_\theta(k)$ and $\hat{\Theta}(k)$ with (A 15), and $\hat{U}(k)$ (as used in (A 2) only) with

$$\hat{U}(k) \equiv U(k,t)^* \exp(-\mu''(k)(t-s)). \tag{A 17}$$

We finally note that $\tau'(k,p,q)$ as used in §§4 and 5 is defined by

$$\tau'(k,p,q) \equiv \frac{1-\exp[-(\mu'_k+\mu'_p+\mu''_q)]}{\mu'_k+\mu'_p+\mu''_q}. \tag{A 18}$$

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